(41), we find the corrections for the elements $\Delta a, \Delta h, \Delta l, \Delta\Omega, \Delta i$, and $\Delta\lambda_0$

The correction for the mean motion is easily found according to the formula

$$\Delta n = -\frac{3}{2} n \frac{\Delta a}{a} \tag{42}$$

The longitude in the orbit λ at time t is determined according to the formula

$$\lambda = (\lambda_0 + \Delta\lambda_0) + (n + \Delta n)t + \delta\lambda \tag{43}$$

where $\delta\lambda$ must be calculated according to formula (24)

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Estimate of Errors for Approximate Solutions of the Simplest Equations of Gasdynamics

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Introduction

In the present article we propose a new method for obtaining approximate generalized solutions for the simplest system of equations of gas dynamics:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial(p + \rho u^2)}{\partial x} = 0$$

with the equation of state

$$p = A\rho^\gamma \qquad A > 0 \qquad \gamma > 1$$

This method closely resembles the one used in (1)

It is shown that the constructed approximate solutions will approximately satisfy the differential equations in the weak sense, i e, for any sufficiently smooth finite function φ we have

$$\iint_{t>0} \left(\rho \frac{\partial \varphi}{\partial t} - \rho u \frac{\partial \varphi}{\partial x}\right) dxdt + \int_{t=0} \rho_0(x)\,\varphi(x,0)dx \to 0$$

$$\iint_{t>0} \left[\rho u \frac{\partial \varphi}{\partial t} - (p + \rho u^2) \frac{\partial \varphi}{\partial x}\right] dxdt + \int_{t=0} \rho_0(x) u_0(x)\, \varphi(x,0)dx \to 0$$

where $\rho_0(x)$, $u_0(x)$ are given functions (initial conditions) The foregoing sums of integrals tend to zero as a parameter h, characteristic of the accuracy of the system, decreases

We shall show that for positive $\varphi(x,t)$:

$$\iint_{t>0} \left[\rho\left(E + \frac{u^2}{2}\right)\frac{\partial \varphi}{\partial t} - \rho u \left(E + \frac{p}{\rho} + \frac{u^2}{2}\right)\frac{\partial \varphi}{\partial x}\right] dxdt + \int_{t=0} \rho_0 \left[E(\rho_0) + \frac{u_0^2}{2}\right] \varphi(x,0)dx$$

will tend to some nonnegative constant as $h \to 0$ As a result, we can be sure that the limit of the approximate solutions (if it exists) cannot contain any inadmissible discontinuous rarefaction waves

We shall first construct the approximate solutions and show that they are bounded In this first part of our discussion we use weaker restrictions on the equation of state $p = p(\rho)$ In order to formulate these restrictions, we introduce the specific volume $V = 1/\rho$ instead of the density ρ and express the equation of state in the form $p = p(\rho) = p(1/V) = g(V)$

Let $g(V)$ satisfy the following conditions:

1) $g(V)$ is defined for all $V > 0$
2) $g'(V) < 0, g''(V) > 0$
3) $0 \leqslant g(V) \leqslant +\infty$; $g(V) \to +\infty$ as $V \to 0$, $g(V) \to 0$ as $V \to \infty$, $V\sqrt{-g'(V)} \to 0$ as $V \to \infty$
4) $\int^V g(V)dV$ is convergent as $V \to \infty$, so that we may set $E = E(\rho) = E(1/V) = -\int_\infty^V g(V)dV$
5) $\int^V \sqrt{-g'(V)}dV$ is convergent as $V \to \infty$ and divergent as $V \to 0$
6) $-(1/V)\int_\infty^V \sqrt{-g'(V)}dV$ is bounded if V is bounded below by a positive constant

All of these conditions are satisfied if the equation of state is of the form

$$p = A\rho^\gamma \qquad A > 0 \qquad \gamma > 1$$

In Sec 1 we state a number of inequalities resulting from our restrictions on the equation of state The first part of the paper will be based on these inequalities In Sec 7 the inequalities will be sharpened, using $p = A\rho^\gamma$ This refinement will enable us to estimate the variation of the solution, which appears in the error estimate

1 Inequalities

The inequalities which we shall state here involve, besides $p(\rho) = g(V), E(\rho)$, the following functions:

$$c(\rho) = V\sqrt{-g'(V)}$$

$$\Phi(\rho) = -\int_\infty^V \sqrt{-g'(V)}dV$$

Translated from Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki (Journal of Computational Mathematics and Mathematical Physics) 1, no 4, 622–637 (1961) Translated by Alexander Schwartz, New York

It follows from property 5 that the boundedness of $\Phi(\rho)$ implies the boundedness of ρ The forementioned inequalities are of the form

$$\frac{d(c+\Phi)}{d\Phi} = -\frac{V}{2}\frac{g''(V)}{g(V)} > 0 \tag{1}$$

$$\Phi\left(\frac{\int_a^b \rho dx}{\int_a^b dx}\right) \leqslant \frac{\int_a^b \rho\Phi(\rho)dx}{\int_a^b \rho dx} \tag{2}$$

$$\int_a^b \left[\rho E(\rho) + \rho\frac{u^2}{2}\right] dx \geqslant (b-a) \times \left[\left(\frac{\int_a^b \rho dx}{b-a}\right) E\left(\frac{\int_a^b \rho dx}{b-a}\right) + \frac{1}{2}\left(\frac{\int_a^b \rho dx}{b-a}\right)\left(\frac{\int_a^b \rho u dx}{\int_a^b \rho dx}\right)^2\right] \tag{3}$$

$$\Phi(\rho_2) - \Phi(\rho_1) < \sqrt{\frac{(\rho_2-\rho_1)[p(\rho_2)-p(\rho_1)]}{\rho_2\rho_1}} \qquad \rho_2 > \rho_1 \tag{4}$$

$$E(\rho_2) - E(\rho_1) + \frac{p(\rho_2)+p(\rho_1)}{2}\left(\frac{1}{\rho_2}-\frac{1}{\rho_1}\right) > 0 \quad \rho_2 > \rho_1 \tag{5}$$

The derivation of these inequalities is based on properties 1–6 listed in the Introduction and on the convexity of the functions involved in them

2 Shock Waves

We shall briefly review those properties of shock waves which are needed in the following discussion

Shock waves are the simplest generalized solutions of the system

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho u}{\partial x} = 0$$

$$\frac{\partial\rho u}{\partial t} + \frac{\partial(p+\rho u^2)}{\partial x} = 0$$

which have the form

$$\rho = \rho(x - wt)$$
$$u = u(x - wt)$$

with piecewise constant $p = p(\xi)$, $u = u(\xi)$, $(\xi = x - wt)$:

$$\rho = \begin{cases}\rho_1, & \xi > 0\\ \rho_2, & \xi < 0\end{cases}$$

$$u = \begin{cases}u_1, & \xi > 0\\ u_2 & \xi < 0\end{cases}$$

It is known that certain relations exist among ρ_1, ρ_2, u_1, u_2 We must also consider the condition for the dissipation of energy in the waves:

$$\iint_{t>0}\left[\rho\left(E+\frac{u^2}{2}\right)\frac{\partial\varphi}{\partial t} - \rho u\left(E+\frac{p}{\rho}+\frac{u^2}{2}\right)\frac{\partial\varphi}{\partial x}\right]dtdx + \int_{t=0}\rho_0\left[E(\rho_0)+\frac{u_0^2}{2}\right]\varphi(x,0)dx \geqslant 0$$

The relations satisfied by the waves are the following:

If $\rho_2 < \rho_1$, then

$$u_1 - u_2 + \sqrt{\frac{(p_2-p_1)(\rho_2-\rho_1)}{\rho_1\ \rho_2}} = 0$$

$$w = u_2 - \frac{1}{\rho_2}\sqrt{\frac{\rho_1\rho_2(p_2-p_1)}{\rho_2-\rho_1}}$$

If $\rho_2 > \rho_1$, then

$$u_1 - u_2 + \sqrt{\frac{(p_2-p_1)(\rho_2-\rho_1)}{\rho_1\rho_2}} = 0$$

$$w = u_2 + \frac{1}{\rho_2}\sqrt{\frac{\rho_1\rho_2(p_2-p_1)}{\rho_2-\rho_1}}$$

If we use the inequality

$$0 < \Phi(\rho_2) - \Phi(\rho_1) < \sqrt{\frac{(p_2-p_1)(\rho_2-\rho_1)}{\rho_1\rho_2}}$$

then it follows from the foregoing relations that regardless of whether $\rho_2 > \rho_1$ or $\rho_2 < \rho_1$, the inequalities

$$\Phi(\rho_2) + u_2 > \Phi(\rho_1) + u_1$$
$$\Phi(\rho_2) - u_2 < \Phi(\rho_1) - u_1$$

are always satisfied In other words $\Phi + u$ is always greater to the left of the wave than to the right, and $\Phi - u$ is always less to the left of the wave than to the right

3 Centered Rarefaction Waves

The foregoing name is applied to discontinuous similarity solutions of the following form:

$$\rho = \begin{cases}\rho_1 & x - \xi_1 t > 0\\ \rho(x/t) & \xi_2 < x/t < \xi_1\\ \rho_2 & x - \xi_2 t < 0\end{cases}$$

$$u = \begin{cases}u_1 & x - \xi_1 t > 0\\ u(x/t) & \xi_2 < x/t < \xi_1\\ u_2 & x - \xi_2 t < 0\end{cases}$$

In order to find $\rho(\xi)$, $u(\xi)$ we obtain a set of ordinary differential equations On solving them, we find:

for $\rho_2 > \rho_1$:

$$\Phi(\rho_2) + u_2 = \Phi(\rho_1) + u_1$$
$$\Phi(\rho_2) - u_2 > \Phi(\rho_1) - u_1$$
$$\xi = u - c(\rho)$$

and for $\rho_2 < \rho_1$:

$$\Phi(\rho_2) - u_2 = \Phi(\rho_1) - u_1 = \Phi(\rho) - u$$
$$\Phi(\rho_2) + u_2 < \Phi(\rho_1) + u_1$$
$$\xi = u + c(\rho)$$

We see that the quantities $\Phi - u$, $\Phi + u$ in the rarefaction wave either remain unchanged or else change in a direction opposite to the one observed in the shock waves Within the rarefaction waves $\Phi + u$, $\Phi - u$ vary monotonically No energy is dissipated in the rarefaction waves

4 The Problem of the Decay of an Arbitrary Discontinuity

We shall construct a generalized solution of the Cauchy problem with the special piecewise constant initial conditions

$$u(x,0) = \begin{cases}u_1, & x > 0\\ u_2, & x < 0\end{cases} \qquad \rho(x,0) = \begin{cases}\rho_1, & x > 0\\ \rho_2, & x < 0\end{cases}$$

$$\rho_1 > 0,\ \rho_2 > 0$$

for arbitrary u_1, u_2, ρ_1, ρ_2 Such a solution may be put together from shock waves and rarefaction waves, that is, from the elements studied in the preceding sections

We first try to find a solution for $\rho > 0$, of the following system of two equations:

$$u = u_1 = \begin{cases} \sqrt{\dfrac{[p(\rho) - p(\rho_1)](\rho - \rho_1)}{\rho\rho_1}} & \rho > \rho_1 \\ \Phi(\rho) - \Phi(\rho_1) & \rho < \rho_1 \end{cases}$$

$$u - u_2 = \begin{cases} -\sqrt{\dfrac{[p(\rho) - p(\rho_2)](\rho - \rho_1)}{\rho\rho_2}} & \rho > \rho_2 \\ \Phi(\rho_2) - \Phi(\rho) & \rho < \rho_2 \end{cases}$$

It can easily be checked that this system is insoluble only if

$$u_2 - u_1 + \Phi(\rho_2) + \Phi(\rho_1) < 0$$

In every case, consequently, we can either make sure that this condition is satisfied or else find the solution of the system One of the following cases can occur:

I) $u_2 - u_1 + \Phi(\rho_2) + \Phi(\rho_1) < 0$

$$\text{II)} \begin{cases} u - u_1 = \sqrt{\dfrac{[p(\rho) - p(\rho_1)(\rho - \rho_1)}{\rho\rho_1}} \\ u - u_2 = -\sqrt{\dfrac{[p(\rho) - p(\rho_2)](\rho - \rho_2)}{\rho\rho_2}} \\ \rho > \rho_1 \qquad \rho > \rho_2 \end{cases}$$

$$\text{III)} \begin{cases} u - u_1 = \sqrt{\dfrac{[p(\rho) - p(\rho_1)](\rho - \rho_1)}{\rho\rho_1}} \\ u - u_2 = \Phi(\rho_2) - \Phi(\rho) \\ \rho_2 > \rho > \rho_1 \end{cases}$$

$$\text{IV)} \begin{cases} u - u_1 = \Phi(\rho) - \Phi(\rho_1) \\ u - u_2 = -\sqrt{\dfrac{[p(\rho) - p(\rho_2)](\rho - \rho_2)}{\rho\rho_2}} \\ \rho_2 < \rho < \rho_1 \end{cases}$$

$$\text{V)} \begin{cases} u - u_1 = \Phi(\rho) - \Phi(\rho_1) \\ u - u_2 = \Phi(\rho_2) - \Phi(\rho) \\ \rho < \rho_1 \qquad \rho < \rho_2 \end{cases}$$

For each of these cases we shall show a method of constructing a solution of the problem of the decay of a discontinuity

Case I

We first find

$$\xi_1 = u_1 + c(\rho_1) \qquad \xi_1^* = u_1 - \Phi(\rho_1)$$

$$\xi_2^* = u_2 + \Phi(\rho_2) \qquad \xi_2 = u_2 - c(\rho_2)$$

$$\xi_1 > \xi_1^* > \xi_2^* > \xi_2$$

After this we define the solution $u(x,t)$, $\rho(x,t)$ in an appropriate manner in each of the following five regions:

1) If $x > \xi_1 t$, we have $u = u_1$, $\rho = \rho_1$

2) If $\xi_1^* t < x < \xi_1 t$, we find u,ρ from the equations

$$\Phi(\rho) - u = \Phi(\rho_1) - u_1 \qquad u + c(\rho) = x/t$$

3) If $\xi_2^* t < x < \xi_1^* t$, we have $\rho = 0$, $u = x/t$ (If $\rho = 0$, we can define u arbitrarily, since it does not appear in the equation independently but is always contained in expressions of the type ρu, ρu^2)

4) If $\xi_2 t < x < \xi_2^* t$, we find u and ρ from the equations

$$\Phi(\rho) + u = \Phi(\rho_2) + u_2$$

$$u - c(\rho) = x/t$$

5) If $x < \xi_2 t$, we have $u = u_2$, $\rho = \rho_2$

In this case the solution consists of two rarefaction waves separated by a vacuum region ($\rho = 0$)

Case II

We first find

$$\xi_1 = u + \frac{1}{\rho}\sqrt{\frac{\rho\rho_1[p(\rho) - p(\rho_1)]}{\rho - \rho_1}}$$

$$\xi_2 = u - \frac{1}{\rho}\sqrt{\frac{\rho\rho_2[p(\rho) - p(\rho_2)]}{\rho - \rho_2}}$$

$$\xi_1 > \xi_2$$

After this we define the solution $u(x,t)$, $\rho(x,t)$ in an appropriate manner in each of the following three regions:

1) If $x > \xi_1 t$

$$u(x,t) = u_1 \qquad \rho(x,t) = \rho_1$$

2) If $\xi_2 t < x < \xi_1 t$

$$u(x\ t) = u \qquad \rho(x,t) = \rho$$

3) If $x < \xi_2 t$:

$$u(x\ t) = u_2 \qquad \rho(x,t) = \rho_2$$

In this case the solution consists of two shock waves—$x = \xi_1 t$, $x = \xi_2 t$ and three constant-valued regions

Case III

We first find

$$\xi_1 = u + \frac{1}{\rho}\sqrt{\frac{\rho\rho_1[p(\rho) - p(\rho_1)]}{\rho - \rho_1}}$$

$$\xi_2^* = u - c(\rho) \qquad \xi_2 = u_2 - c(\rho_2)$$

$$\xi_2 < \xi_2^* < \xi_1$$

We then define the solution $u(x,t)$, $\rho(x,t)$ in an appropriate manner in each of the following four regions:

1) If $x > \xi_1 t$:

$$u(x,t) = u_1 \qquad \rho(x,t) = \rho_1$$

2) If $\xi_2^* t < x < \xi_1 t$

$$u(x\ t) = u \qquad \rho(x,t) = \rho$$

3) If $\xi_2 t < x < \xi_2^* t$, we find $u(x,t)$, $\rho(x,t)$ from the equations

$$\Phi[\rho(x,t)] + u(x,t) = \Phi(\rho_2) + u_2$$

$$u(x,t) - c[\rho(x\ t)] = x/t$$

4) If $x < \xi_2 t$:

$$u(x,t) = u_2 \qquad \rho(x,t) = \rho_2$$

The solution constructed here contains one shock wave, one rarefaction wave, and three constant-valued regions

Case IV

We first find

$$\xi_1 = u_1 + c(\rho_1) \qquad \xi_1^* = u + c(\rho)$$

$$\xi_2 = u - \frac{1}{\rho}\sqrt{\frac{\rho\rho_2[p(\rho) - p(\rho_2)]}{\rho - \rho_2}}$$

We then define the solution $u(x,t)$, $\rho(x,t)$ in an appropriate manner in each of the following four regions:

1) If $x > \xi_1 t$:

$$u(x,t) = u_1 \qquad \rho(x,t) = \rho_1$$

2) If $\xi_1^* t < x < \xi_1 t$, we find $u(x,t)$ $\rho(x,t)$ from the equations

$$\Phi[\rho(x,t)] - u(x,t) = \Phi(\rho_1) - u_1$$

$$u(x,t) + c[\rho(x\ t)] = x(t)$$

3) If $\xi_2 t < x < \xi_1^* t$

$$u(x,t) = u \qquad \rho(x,t) = \rho$$

4) If $\xi_2 t > x$

$$u(x,t) = u_2 \qquad \rho(x,t) = \rho_2$$

The structure of the solution in this case is similar to that in Case III

Case V

We first calculate

$$\xi_1 = u_1 + c(\rho_1) \qquad \xi_1^* = u + c(\rho)$$
$$\xi_2^* = u - c(\rho) \qquad \xi_2 = u_2 - c(\rho_2)$$

We then define the solution in each of the five regions listed next in the following manner:

1) If $x > \xi_1 t$

$$u(x,t) = u_1 \qquad \rho(x,t) = \rho_1$$

2) If $\xi_1^* t < x < \xi_1 t$, we find $u(x,t)$ $\rho(x,t)$ from the equations

$$\Phi[\rho(x,t)] - u(x,t) = \Phi(\rho_1) + u_1$$
$$u(x,t) + c[\rho(x,t)] = x/t$$

3) If $\xi_2^* t < x < \xi_1^* t$:

$$u(x,t) = u \qquad \rho(x,t) = \rho$$

4) If $\xi_2 t > x > \xi_2^* t$, we find $u(x,t)$, $\rho(x,t)$ from the equations

$$\Phi[\rho(x,t)] + u(x,t) = \Phi(\rho_2) + u_2$$
$$u(x,t) - c[\rho(x,t)] = x/t$$

5) If $x < \xi_2 t$

$$u(x,t) = u_2 \qquad \rho(x,t) = \rho_2$$

In this last case the solution contains two rarefaction waves and three constant-valued regions

All of the constructed solutions satisfy the equations and the energy dissipation condition, since they consist of elementary solutions satisfying the equations and the dissipation condition

Making use of the structure of the solution and the properties of shock waves and rarefaction waves, we can readily find that in the case of decay of a discontinuity the values of $\Phi + u$ and $\Phi - u$ do not exceed the greater of the values $\Phi_1 + u_1$, $\Phi_2 + u_2$; $\Phi_1 - u_1$ $\Phi_2 - u_2$ Moreover if $\Phi_1 + u_1 < M$, $\Phi_2 + u_2 < M$, $\Phi_1 - u_1 < M$, $\Phi_2 - u_2 < M$, then there exists a positive number $\zeta = \zeta(M)$ depending only on M such that if $|x/t| < \zeta$, the values of u and ρ will always coincide with the initial conditions (with u_1, ρ_1 if $x/t > \zeta$ and with u_2, ρ_2 if $x/t < -\zeta$) In fact, since the maximum values of $\Phi + u$, $\Phi - u$ do not increase in the case of decay of a discontinuity, it follows that we have $\Phi + u < M$ $\Phi - u < M$ for the entire solution Assuming that $\Phi \geqslant 0$, we have

$$|u| < M \qquad |\Phi| < 2M$$

We have obtained an upper estimate for u and ρ which depends only on M Hence we can readily find an estimate for the quantities ξ_1, ξ_2 characteristic of the velocity of propagation of the disturbances For this purpose we must carefully examine all the cases and find a simple estimate for each It is clear that an estimate for ξ_1, ξ_2 will depend only on M It is sufficient to take the greatest of the possible values of $|\xi_1|$, $|\xi_2|$ as ζ

The solution of the problem of the decay of a discontinuity and the properties of shock waves and rarefaction waves are studied in courses in gas dynamics The present article gives a fairly detailed description of the basic facts and outlines their derivation, in order to call attention to certain details which are usually overlooked

5 Approximate Solutions of the Cauchy Problem

In this section we shall discuss a method for constructing approximate solutions of the Cauchy problem and give simple a priori estimates for them

Let the initial conditions $u(x,0)$, $\rho(x\ 0) > 0$ be bounded and measurable periodic functions with a period of unity In this case the solution also will belong to the class of periodic functions

We shall assume that all functions considered in the interval (0,1) can be periodically continued to the entire real line

We divide the interval (0,1) into N equal parts of length $h = 1/N$ and replace the initial distribution of the quantities u,ρ by step functions which are constant in each interval of length h In order to calculate the values of the step functions we use the formulas

$$\rho h = \frac{1}{h}\int_{(k-1)h}^{kh} \rho(x,0)dx \qquad kh > x > (k-1)h$$

$$\rho_h u_h = \frac{1}{h}\int_{(k-1)h}^{kh} \rho(x,0)u(x,0)dx,\ kh > x > (k-1)h$$

From the boundedness of $\rho(x,0)$, $u(x,0)$ and from the assumption that $\rho(x,0) > \text{const} > 0$, it follows that the functions ρ_h, u_h are bounded by predetermined constants not depending on h We shall assume that

$$\Phi(\rho_h) + u_h < M \qquad \Phi(\rho_h) - u_h < M$$

where M is independent of h

We now proceed to find an approximate solution of the Cauchy problem, using the step functions ρ_h, u_h as the initial conditions It is found that the forementioned problem of the decay of a discontinuity makes it possible to construct an exact solution for such initial conditions if the values of t are not too high

At each boundary between two adjacent steps at $t = 0$ there is a decay of the initial discontinuity, and these discontinuities will not interact until the waves from two adjacent discontinuities meet We found in the preceding section that the velocity of propagation of the disturbances will not exceed $\zeta(M)$ If we consider the time interval $\tau = h/3\zeta(M)$, we see that the disturbance cannot travel during this time more than one-third of the way from each end of a subdivision toward its center, so that one-third will certainly remain undisturbed At $t = \tau$ we average the solution obtained by the method described just as we averaged the initial conditions After this the solution will be piecewise constant, with the same constant-valued intervals We note that ρ will be positive in each interval and $\Phi + u$ and $\Phi - u$ will again be less than or equal to M This follows from the property of discontinuity decay that the maximum of $\Phi + u$, $\Phi - u$ is not increased from the inequality

$$\Phi\left(\frac{\int_{kh}^{(k+1)h} \rho dx}{h}\right) \leqslant \frac{\int_{kh}^{(k+1)h} \rho\Phi(\rho)\,dx}{\int_{kh}^{(k+1)h} \rho dx}$$

derived in Sec 1, and from the fact that

$$\frac{\int_{kh}^{(k+1)h} \rho[\Phi \pm u]dx}{\int_{kh}^{(k+1)h} \rho dx} \leqslant \max(\Phi \pm u)$$

Therefore by repeating beyond time $t = \tau$ the process described for $0 < t < \tau$, we can obtain an approximate solution in the interval $\tau < t < 2\tau$ At $t = 2\tau$ we repeat the averaging process, which will take us to $t = 3\tau$, and so on In $1/\tau = 3\zeta(M)/h$ steps we arrive at time $t = 1$ It should be noted that for this approximate solution there is an a priori estimate:

$$\Phi(\rho) + u < M$$
$$\Phi(\rho) - u < M$$

from which, as we have already seen, it follows that ρ and u are bounded by constants independent of h

6 Estimate of Errors in Finding Approximate Solutions Satisfying Equations

In Sec 5 we constructed approximate solutions for the problem under consideration; each of these solutions is characterized by its parameter value h As h approaches zero, we obtain an infinite family of approximate solutions Each of these solutions consists of a pair ρ_h, u_h of piecewise continuous functions given in the region $0 \leqslant t \leqslant 1, 0 \leqslant x \leqslant 1$, where these functions are uniformly bounded independently of h

We shall show that as h decreases, the functions $\rho_h\, u_h$ will satisfy more and more exactly the equations and the inequality which appear in the definition of the generalized solution Thus we obtain the following estimates:

$$\iint_{t>0} \left(\rho_h \frac{\partial \varphi}{\partial t} - \rho_h u_h \frac{\partial \varphi}{\partial x}\right) dx\, dt + \int_{t=0} \rho_{0h}\varphi(x,0)dx = 0(h^{1/3})$$

$$\iint_{t>0} \left\{\rho_h u_h \frac{\partial \varphi}{\partial t} - [p(\rho_h) + \rho_h u_h^2] \frac{\partial \varphi}{\partial x}\right\} dx\, dt + \int_{t=0} \rho_{0h}\varphi(x,0)dx = 0(h^{1/3})$$

$$\iint_{t>0} \left[\rho_h \left(E_h + \frac{u_h^2}{2}\right) \frac{\partial \varphi}{\partial t} - \rho_h u_h \left(E_h + \frac{p_h}{\rho_h} + \frac{u_h^2}{2}\right) \frac{\partial \varphi}{\partial x}\right] \times dx\, dt + \int_{t=0} \rho_{0h}\left(E_{0h} + \frac{u_{0h}^2}{2}\right) \varphi(x,0)dx \geqslant -\text{const } h^{1/3}$$

(for positive finite values of φ)

Since our method of constructing the functions ρ_{0h}, u_{0h} guarantees that they will tend to ρ_0, u_0 in the weak sense as $h \to 0$, these estimates show that the equations will be satisfied more and more exactly (in the weak sense) as $h \to 0$

In the proposed method the approximate solution is built up from layers within which the gasdynamics equations are satisfied exactly Therefore, within each layer, we have

$$\iint_{t_{k+1}>t>t_k} \left(\rho_h \frac{\partial \varphi}{\partial t} - \rho_h u_h \frac{\partial \varphi}{\partial x}\right) dx\, dt + \int_{t=t_k} \rho_h^+(x,t_k)\, \varphi(x,t_k)dx - \int_{t=t_{k+1}} \rho_h^-(x,t_{k+1})\, \varphi(x,t_{k+1})dx = 0$$

$$\iint_{t_{k+1}>t>t_k} \left[\rho_h u_h \frac{\partial \varphi}{\partial t} - (p_h + \rho_h u_h^2) \frac{\partial \varphi}{\partial x}\right] dx\, dt + \int_{t=t_k} \rho_h^+ u_h^+ \varphi dx - \int_{t=t_{k+1}} \rho_h^- u_h^- \varphi\, dx = 0$$

$$\iint_{t_{k+1}>t>t_k} \left[\rho_h \left(E_h + \frac{u_h^2}{2}\right) \frac{\partial \varphi}{\partial t} - \rho_h u_h \left(E_h + \frac{p_h}{\rho_h} + \frac{u_h^2}{2}\right)\frac{\partial \varphi}{\partial x}\right] dx\, dt + \int_{t=t_k} \rho_h^+ \times \left[E_h^+ + \frac{(u_h^+)^2}{2}\right] \varphi\, dx - \int_{t=t_{k+1}} \rho_h^- \times \left[E_h^- + \frac{(u_h^-)^2}{2}\right] \varphi\, dx \geqslant 0$$

where we use the notation

$$\rho_h^+(x,t_k) = \rho_h(x,t_k + 0) \qquad u_h^+(x,t_k) = u_h(x,t_k + 0)$$

$$\rho_h^-(x,t_k) = \rho_h(x,t_k - 0) \qquad u_h^-(x,t_k) = u_h(x,t_k - 0)$$

Hence it can be seen that the problem reduces to an estimate of the sums

$$\sum_k \int_{t=t_k} (\rho_h^+ - \rho_h^-)\varphi dx$$

$$\sum_k \int_{t=t_k} (\rho_h^+ u_h^+ - \rho_h^- u_h^-)\varphi dx$$

$$\sum_k \int_{t=t_k} \left\{\rho_h^+ \left[E_h^+ + \frac{(u_h^+)^2}{2}\right] - \rho_h^- \left[E_h^- + \frac{(u_h^-)^2}{2}\right]\right\} \varphi dx$$

In order to estimate these sums we must recall how we obtain ρ_h^+ and ρ_h^+, u_h^+ from ρ_h^-, u_h^- In the preceeding section they were obtained as the result of an averaging process Let us recall, for example, the formula for ρ_h^+:

$$\rho_h^+ = \frac{1}{h}\int_{ph}^{(p+1)h} \rho_h^-\, dx \qquad (p + 1)h > x > ph$$

We shall prove in the following discussion that for the sum of the variations of the function $\rho_h^-(x,t_k)$ over all layers $t = t_k$ the inequality

$$\sum_k \int_{t=t_k} |d\rho_h^-| \leqslant \frac{A}{h^{5/3}}$$

holds Inequalities of exactly the same kind will be proved for the functions ρ_h^-, u_h^-, $\rho_h^+[E_h^+ + (u_h^+)^2/2]$ Moreover, we shall show that if φ is a sufficiently smooth finite periodic function and $\bar{z}$ is obtained from a bounded z by an averaging process using the interval h, then

$$\left|\int_0^1 \varphi(x)(z - \bar{z})dx\right| \leqslant Bh^2 \int_0^1 |dz| + Dh^2$$

From these two statements it follows immediately that

$$\sum_k \int_{t=t_k} (\rho_h^+ - \rho_h^-)\varphi dx = O(h^{1/3})$$

$$\sum_k (\rho_h^+ u_h^+ - \rho_h^- u_h^-)\varphi dx = O(h^{1/3})$$

The estimate for

$$\sum_k \int_{t=t_k} \left\{\rho_h^+ \left[E_h^+ + \frac{(u_h^+)^2}{2}\right] - \rho_1^- \left[E_h^- + \frac{(u_h^-)^2}{2}\right]\right\} \varphi\, dx$$

is obtained in a somewhat more complicated manner, since $\rho_h^+[E_h^+ + (u_h^+)^2/2]$ is not obtained explicitly by averaging $\rho_h^-[E_h^- + (u_h^-)^2/2]$ but is expressed by the quantities ρ_h^+, u_h^+ obtained by averaging For simplicity of notation, we shall write the functions ρ_h^+, u_h^+ simply as $\rho(x), u(x)$ Instead of ρ_h^+, u_h^+ we shall write simply $\bar{\rho}, \bar{u}$ The quantity $\rho_h{}^+[E_h^+ + (u_h^+)^2/2]$ will be designated by $\bar{\rho}[E(\bar{\rho}) + \bar{u}^2/2]$, in contradistinction to the quantity $\overline{[\rho E + \rho u^2/2]}$, which is obtained by direct averaging of the expression within the brackets From inequality (3) of Sec 1:

$$\int_a^b \left[\rho E(\rho) + \rho\frac{u^2}{2}\right] dx \geqslant (b - a) \times \left[\left(\frac{\int_a^b \rho dx}{b - a}\right) E\left(\frac{\int_a^b \rho dx}{b - a}\right) + \frac{1}{2}\left(\frac{\int_a^b \rho dx}{b - a}\right)\left(\frac{\int_a^b \rho u\, dx}{\int_a^b \rho dx}\right)^2\right]$$

it follows that

$$\overline{\left[\rho E + \frac{\rho u^2}{2}\right]} \geqslant \bar{\rho}\left[E(\bar{\rho}) + \frac{\bar{u}^2}{2}\right]$$

Therefore, for positive values of φ we have

$$\sum_k \int_{t=t_k} \varphi\left\{\bar{\rho}\left[E(\bar{\rho}) + \frac{\bar{u}^2}{2}\right] - \overline{\left[\rho E + \frac{\rho u^2}{2}\right]}\right\} dx \leqslant 0$$

The estimate for the sum

$$\sum_k \int_{t=t_k} \varphi\left\{\overline{\left[\rho E + \frac{\rho u^2}{2}\right]} - \left[\rho E(p) + \frac{\rho u^2}{2}\right]\right\} dx \leqslant Ah^{1/3}$$

is obtained in the same way as the estimate for the sums containing the integrals of ρ, u

Adding the last two inequalities, we obtain

$$\sum_k \int_{t=t_k} \varphi \left\{ \bar{\rho} \left[E(\bar{\rho}) + \frac{\bar{u}^2}{2} \right] - \rho \left[E(\rho) + \frac{\rho u^2}{2} \right] \right\} dx \leqslant Ah^{1/3}$$

It follows that

$$\iint_{t>0} \left[\rho_h \left(E_h + \frac{u_h^2}{2} \right) \frac{\partial \varphi}{\partial t} - \rho_h u_h \left(E_h + \frac{p_h}{\rho_h} + \frac{u_h^2}{2} \right) \frac{\partial \varphi}{\partial x} \right] \times dx\, dt + \int_{t=0} \rho_h \left[E_h + \frac{u_h^2}{2} \right] \varphi dx \geqslant -Ah^{1/3}$$

Next we shall prove the estimate

$$\left| \int_0^1 \varphi(x)(z - \bar{z})\, dx \right| \leqslant Bh^2 \int_3^1 |dz| + Dh^2$$

for bounded z with bounded variation and sufficiently smooth φ Let us recall that $z(1) = z(0)$ and

$$\bar{z}(x) = \frac{1}{h} \int_{ph}^{(p+1)h} z(x) dx \qquad ph < x < (p+1)h$$

Let $\varphi(x)$ have bounded second derivatives Then in the interval $[ph, (p+1)h]$ we have the representation

$$\varphi(x) = a_p + b_p[x - (p + \tfrac{1}{2})h] + [x - (p + \tfrac{1}{2})h]^2 c_p(x)$$

where $c_p(x)$ is a bounded function and $b_p = \varphi'[(p + \frac{1}{2})h]$ We have

$$\int_{ph}^{(p+1)h} [z(x) - \bar{z}(x)]\{a_p + b_p[x - (p + \tfrac{1}{2})h] + [x - (p + \tfrac{1}{2})h]^2 c_p(x)\} dx = a_p \int_{ph}^{(p+1)h} [z(x) - \bar{z}(x)] dx + b_p \int_{ph}^{(p+1)h} [z(x) - \bar{z}(x)][x - (p + \tfrac{1}{2})h] dx + \int_{ph}^{(p+1)h} [z(x) - \bar{z}(x)][x - (p + \tfrac{1}{2})h]^2 c_p(x) dx$$

It is readily seen that

$$\int_{ph}^{(p+1)h} [z(x) - \bar{z}(x)] dx = 0$$

$$\left| \int_{ph}^{(p+1)h} [z(x) - \bar{z}(x)] \left[x - \left(p + \frac{1}{2} \right) h \right]^2 c_p(x) dx \right| < D_1 h^3$$

Thus, all that remains is to calculate the integral for b_p Integrating by parts, we obtain

$$\int_{ph}^{(p+1)h} [z(x) - \bar{z}(x)][x - (p + \tfrac{1}{2})h] dx = \frac{z[(p+1)h] - z(ph)}{8} h^2 - \int_{ph}^{(p+1)h} \frac{[x - (p + \frac{1}{2})h]^2}{2} dz$$

$$\left| \int_{ph}^{(p+1)h} \frac{[x - (p + \frac{1}{2})h]^2}{2} dz \right| \leqslant \frac{h^2}{8} \int_{ph}^{(p+1)h} |dz|$$

Then

$$\left| \int_{ph}^{(p+1)h} [z(x) - \bar{z}(x)] \varphi(x) dx - \varphi'\left[\left(p + \frac{1}{2} \right) h \right] \times \frac{z[(p+1)h] - z(ph)}{8} h^2 \right| \leqslant D_1 h^3 + Bh^2 \int_{ph}^{(p+1)h} |dz|$$

Adding such estimates for all the intervals, we obtain

$$\left| \int_0^1 [z(x) - \bar{z}(x)] \varphi(x) dx - \frac{h^2}{8} \sum_p \varphi'\left[\left(p + \frac{1}{2} \right) h \right] \times \{z[(p+1)h] - z(ph)\} \right| \leqslant D_1 h^2 + B$$

We estimate the sum

$$\Sigma_p\, \varphi'[(p + \tfrac{1}{2})h]\{z[(p+1)h] - z(ph)\} = -\Sigma_p\, z(ph)\{\varphi'[(p + \tfrac{1}{2})h] - \varphi'[(p - \tfrac{1}{2})h]\}$$

Since $z(ph)$ is bounded, $|\varphi'[(p + \frac{1}{2})h] - \varphi'[(p - \frac{1}{2})h]| <$ const h, and the number of terms is equal to $1/h$, it follows that

$$\left| \sum \varphi'[(p + \tfrac{1}{2})h]\{z[(p+1)h] - z(ph)\} \right| \leqslant D_2$$

Now it is clear that

$$\left| \int_0^1 [z(x) - \bar{z}(x)] \varphi(x) dx \right| \leqslant Bh^2 \int_0^1 |dz| + Dh^2$$

Thus we have completed the process of reducing the estimate of the error to an estimate of the variation of the approximate solutions

7 Refinement of Certain Inequalities from Section 1

In Sec 1 we used the inequalities

$$\Phi(\rho_2) - \Phi(\rho_1) < \sqrt{\frac{(\rho_2 - \rho_1)[p(\rho_2) - p(\rho_1)]}{\rho_1 \rho_2}}$$

$$E(\rho_2) - E(\rho_1) + \frac{p(\rho_2) + p(\rho_1)}{2} \left(\frac{1}{\rho_2} - \frac{1}{\rho_1} \right) < 0 \qquad \rho_2 > \rho_1$$

We now wish to refine these inequalities somewhat, using a concrete form of the equation of state $p = A\rho\gamma (A > 0$ $\gamma > 1)$; specifically, we shall show that

$$\Phi(\rho_2) - \Phi(\rho_1) < \sqrt{\frac{(\rho_2 - \rho_1)[p(\rho_2) - p(\rho_1)]}{\rho_2 \rho_1}} < \Phi(\rho_2) - \Phi(\rho_1) + \nu \frac{[\Phi(\rho_2) - \Phi(\rho_1)]^3}{[\Phi(\rho_2) + \Phi(\rho_1)]^2}$$

$$E(\rho_2) - E(\rho_1) + \frac{p(\rho_2) + p(\rho_1)}{2} \left(\frac{1}{\rho_2} - \frac{1}{\rho_1} \right) < -\varkappa \sqrt{\frac{\rho_2 - \rho_1}{\rho_1 \rho_2 [p(\rho_2) - p(\rho_1)]}} \frac{[\Phi(\rho_2) - \Phi(\rho_1)]^3}{[\Phi(\rho_2) + \Phi(\rho_1)]^{2\gamma/(\gamma-1)}}$$

(ν and $\varkappa$ are positive constants)

We shall begin the proof by expressing E, p, ρ in terms of Φ:

$$E = \frac{4\gamma A^2}{(\gamma - 1)^3} \Phi^2$$

$$p = A \left[\frac{2\sqrt{\gamma A}}{\gamma - 1} \right]^{2\gamma/(\gamma-1)} \Phi^{2\gamma/(\gamma-1)}$$

$$\rho = \left[\frac{2\sqrt{\gamma A}}{\gamma - 1} \right]^{2/(\gamma-1)} \Phi^{2/(\gamma-1)}$$

We shall prove the inequality

$$\sqrt{\frac{(\rho_2 - \rho_1)(p_2 - p_1)}{\rho_2 \rho_1}} < \Phi_2 - \Phi_1 + \nu \frac{[\Phi_2 - \Phi_1]^3}{[\Phi_2 + \Phi_1]^2} \qquad (\Phi_2 > \Phi_1)$$

We rewrite it in the form

$$F = \frac{\left[\sqrt{\frac{(\rho_2 - \rho_1)(p_2 - p_1)}{\rho_2 \rho_1}} - (\Phi_2 - \Phi_1) \right] (\Phi_2 + \Phi_1)^2}{[(\Phi_2 - \Phi_1]^3} < \nu$$

If we express ρ and p in terms of Φ, it can be shown that the left side of the inequality is a homogeneous function of

Φ_2, Φ_1, dependent essentially only on

$$\eta = \frac{\Phi_2 - \Phi_1}{\Phi_2 + \Phi_1} > 0$$

The expression is undefined at $\eta = 0$, but it can be shown that lim $F(\eta)$ is finite The finiteness of lim $F(\eta)$ follows from the fact that the numerator and denominator are of the same degree in η When $\eta \neq 0$, the denominator does not vanish From this it follows that $F(\eta)$ is bounded:

$$F(\eta) < v$$

The second inequality is proved in a completely similar manner

8 Estimate of the Variation of Approximate Solutions

We now proceed to find an estimate for the variation of an approximate solution

Let us consider some layer $t_{k+1} > t > t_k$, within which our approximate solution is an exact solution of the equation of gasdynamics, consisting of alternating shock waves and rarefaction waves The amplitude of each of these waves is constant during the time interval considered and consequently the integrals which must be estimated,

$$\int_0^1 |d(u + \Phi)| \qquad \int_0^1 |d(\Phi - u)|$$

are also constant Judging by the properties of shock waves and rarefaction waves (see Secs 2 and 3), $\Phi + u$ will increase from right to left in shock waves and decrease from right to left in rarefaction waves Since $\Phi + u$ is a periodic function, the sum of the amplitudes of the shock waves (numerically equal to $\Phi + u$) is equal to the sum of the amplitudes of the rarefaction waves Consequently the total variation of $\Phi + u$ is equal to twice the sum of the jumps of this quantity in shock waves

A similar discussion applies to the variation of the quantity $\Phi - u$

Let us now try to estimate the sum of the absolute values of the jumps in $\Phi + u$, $\Phi - u$ by means of the jumps in Φ in some single wave We take the case of a wave for which ρ_2 (ρ to the left of the wave) is greater than ρ_1(ρ to the right of the wave) The contrary case is considered in a similar manner

We know that if $\rho_2 > \rho_1$, then

$$u_1 - u_2 + \sqrt{\frac{(p_2 - p_1)(\rho_2 - \rho_1)}{\rho_1\rho_2}} = 0 \qquad (6)$$

Using our inequality from Sec 7, we obtain

$$u_1 - u_2 + \Phi_2 - \Phi_1 + \nu \frac{(\Phi_2 - \Phi_1)^3}{(\Phi_2 + \Phi_1)^2} > 0$$

which is equivalent to the statement

$$|(\Phi_1 - u_1) - (\Phi_2 - u_2)| = (\Phi_1 - u_1) - (\Phi_2 - u_2) < \nu \frac{(\Phi_2 - \Phi_1)^3}{(\Phi_2 + \Phi_1)^2}$$

Equation (6) may be rewritten as follows:

$$u_1 - u_2 + \Phi_1 - \Phi_2 = -(\Phi_2 - \Phi_1) - \sqrt{\frac{(p_2 - p_1)(\rho_2 - \rho_1)}{\rho_1\rho_2}} > -2(\Phi_2 - \Phi_1) - \nu \frac{(\Phi_2 - \Phi_1)^3}{(\Phi_2 + \Phi_1)^2}$$

Hence

$$|(\Phi_1 + u_1) - (\Phi_2 + u_2)| = (\Phi_2 + u_2) - (\Phi_1 + u_1) < 2(\Phi_2 - \Phi_1) + \nu \frac{(\Phi_2 - \Phi_1)^3}{(\Phi_2 + \Phi_1)^2}$$

Finally, for jumps in the shock wave we have the inequality

$$|\Delta(\Phi + u)| + |\Delta(\Phi - u)| \leqslant 2|\Delta\Phi| + 2\nu \frac{|\Delta\Phi|^3}{\sigma^2}$$

$$\sigma = \Phi_2 + \Phi_1$$

Thus, the estimate of the variation has been reduced to the estimate of the jumps in Φ in shock waves

In order to estimate the jumps in Φ we use the following reasoning: In each shock wave the dissipation of energy is proportional to the cube of the wave amplitude $|\Delta\Phi|^3$ and to the time interval τ during which the wave exists The sum of the energies dissipated in the waves cannot exceed the initial supply of energy From this we obtain the inequality

$$\tau \sum \sqrt{\frac{\rho_1\rho_2[p_2 - p_1]}{\rho_2 - \rho_1}} \left| E_2 - E_1 + \frac{p_2 + p_1}{2} \times \left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right)\right| \leqslant \int_{t=0} \rho \left(E + \frac{u^2}{2}\right) dx \leqslant M'$$

Using the inequality from Sec 7 we can also rewrite this as

$$\tau \sum \frac{|\Delta\Phi|^3}{\sigma^{2\gamma/(\gamma-1)}} < \frac{M'}{\varkappa}$$

It follows from this that

$$\sum \frac{|\Delta\Phi|^3}{\sigma^2} = \sum \sigma^{2/(\gamma-1)} \frac{|\Delta\Phi|^3}{\sigma^{2\gamma/(\gamma-1)}} \leqslant \frac{M''M'}{\tau\varkappa} < \frac{\text{const}}{k}$$

$$\sum |\Delta\Phi| \leqslant \sqrt[3]{\sum \sigma^{2\gamma/(\gamma-1)} \frac{|\Delta\Phi|^3}{\sigma^{2\gamma/(\gamma-1)}}} \times (\text{no of waves})^{2/3} \leqslant \text{const}\, \frac{1}{\sqrt[3]{h}} \left(\frac{1}{h^2}\right)^{2/3} = \frac{\text{const}}{h^{5/3}}$$

Clearly

$$\sum \left(2|\Delta\Phi| + 2\nu \frac{|\Delta\Phi|^3}{\sigma^2}\right) \leqslant \frac{\text{const}}{h} + \frac{\text{const}}{h^{5/3}} < \frac{\text{const}}{h^{5/3}}$$

and

$$\Sigma|\Delta(\Phi + u)| + \Sigma|\Delta(\Phi - u)| < \text{const}/h^{5/3}$$

We have explained how this estimate provides us with an estimate of the variations of $\Phi + u$, $\Phi - u$, and consequently of the quantities Φ, u, as well Since all other quantities appearing in the problem can be represented as functions of Φ,u with bounded derivatives in a bounded region (the boundedness of Φ,u has been proved), it follows directly that we have also found the necessary variation estimate for all of these quantities

—Submitted March 4, 1961

Reference

[1] Godunov, S K, "A difference method for the numerical calculation of discontinuous solutions of the equations of hydrodynamics," Matemat Sbornik (Math Symp) **47** (89), no 3, 271–306 (1959)

Reviewer's Comments

The author presents a method for obtaining a solution to one-dimensional hydrodynamic flow The hydrodynamic material is divided into zones in which the parameters are considered initially constant The method of characteristics is used to obtain the solution of the hydrodynamic equations that apply between adjacent zones until a disturbance has entered one-third the zone thickness from either side of a zone At this time the solution is remapped to another grid where once again the parameters are initially constant in a zone The remapping procedure conserves mass and momentum A library of solution is made ahead of time for the possible combinations of shock and rarefactions that would be possible between two adjacent zones

The method is applied to a single fluid with an equation of state where the pressure is a function of density alone The technique could be readily extended to a two-fluid problem However, to use an equation of state that had an energy dependence would add a considerable complication to the calculation

In general, the method appears to be a practical way of constantly rezoning a hydrodynamic calculation in Eulerion coordinates It is especially suited to the problem of a gas expanding into a vacuum In principal the technique could also be applied to two-dimensional calculations

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Free Molecular Heat Transfer near a Wall

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IN heat transfer between a gas and a wall, the heat flux can be determined by considering the transfer of heat at the wall through a layer so thin that the chances of intermolecular collisions can be disregarded

Blodgett and Langmuir[1] have analyzed heat transfer between two coaxial cylinders under the following conditions Heat from the inside cylinder at a temperature T is transferred by "free" molecules over a distance equal to the mean free path l, where the temperature is T'_a (the difference $T - T_a'$ is assumed to be the temperature drop) The heat is then transferred in accordance with the usual laws of heat conduction

Without imposing limitations due to the possible effect of curvature of the cylinder, the investigators arrived at the following equation for the heat flux:

$$g = a\left(\beta + \frac{1}{2}\right) pk \frac{1}{\sqrt{2\pi mkT'_a}} (T - T_a') \tag{1}$$

where a is the accommodation coefficient on the surface of the inside cylinder, p the pressure, k the Boltzmann constant, m the mass of the molecule, and βk the specific heat per molecule at constant volume, in erg/K

The number of molecules incident on a surface area of 1 cm^2/sec is

$$N = \tfrac{1}{4} n\bar{v} \tag{2}$$

where n is the number of molecules per unit volume, and v is their arithmetic mean velocity

Accordingly, we can transform Eq (1) as follows:

$$g = a\Lambda_0 \rho \sqrt{273\ 2/T_a}(T - T_a') \tag{3}$$

where Λ_0 is the thermal conductivity of the "free" molecules in erg/sec cm^2 deg bar:

$$\Lambda_0 = \frac{1}{2}\left(\frac{\gamma+1}{\gamma-1}\right)\sqrt{\frac{R}{2\pi}} \frac{1}{\sqrt{M(273\ 2)}} \tag{4}$$

where $\gamma = c_p/c$ is the ratio of heat capacities

Kyte, Madden, and Piret[2] employ Eq (3) in determining the limits of the effect of free molecular transfer on heat exchange at low pressures, and also in calculating the heat flux

Deshman[3] obtained an equation of type (3) for free molecular transfer between two infinite plates Instead of a he used a reduced accommodation coefficient

$$a_{\text{educ d}} = a_1 a_2/(a_1 + a_2 - a_1 a_2) \tag{5}$$

where a_1 and a_2 are the accommodation coefficients on the surface of the plates

Equation (3) represents a case of heat transfer between plates in which the accommodation coefficient on one of the plates is equal to unity This means that if Eq (3) is to remain valid for the given heat transfer conditions, we must assume the accommodation coefficient to be equal to unity on the gas side

Blodgett and Langmuir do in fact make this assumption in their physical model, assuming that the temperature of the molecules incident on the surface is T_a But T_a is the average temperature of the molecules incident on the wall from the gas side and moving away from the wall The number of molecules with opposite velocity components is equal; consequently, molecules at a temperature higher than T_a are incident on the wall Hence, the heat flux should not be expressed by Eq (3)

In our analysis we will assume that the temperature drop at the surface is $T - T_a$, where T is the surface temperature and T_a (contrary to the premises adopted in Ref 1) is the temperature of the gas adjacent to the wall This corresponds to the formulation of the boundary conditions in problems of rarefied gas mechanics

Let us compare the free molecular heat transfer near a plane horizontal wall with the heat conduction far from it

Let us select a reference surface far from the wall Let the temperature of the gas on both sides of the selected plane, at a distance of the mean free path from the plane, be T'' and T', respectively It is also assumed that $T'' > T'$

The heat flux through the reference surface is

$$g = -\lambda \Delta T/\Delta x \tag{6}$$

where λ is the thermal conductivity, $\Delta T/\Delta x$ is the temperature gradient, which can usually be expressed[4] in the following terms:

$$\Delta T/\Delta x = (T'' - T')/2\bar{l} \tag{7}$$

When the difference $T'' - T'$ is small, the number of molecules moving up through the reference plane is equal to the number of molecules moving down through it This is because the energy of the molecules incident on the plane from both sides is equally and incrementally higher in absolute

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